On the H-Free Extension Complexity of the TSP

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Abstract

It is known that the extension complexity of the TSP polytope for the complete graph K_n is exponential in n even if the subtour inequalities are excluded. In this article we study the polytopes formed by removing other subsets $\mathcal{H}$ of facet-defining inequalities of the TSP polytope. In particular, we consider the case when $\mathcal{H}$ is either the set of blossom inequalities or the simple comb inequalities. These inequalities are routinely used in cutting plane algorithms for the TSP. We show that the extension complexity remains exponential even if we exclude these inequalities. In addition we show that the extension complexity of polytope formed by all comb inequalities is exponential. For our proofs, we introduce a subclass of comb inequalities, called (h, t)-uniform inequalities, which may be of independent interest.

Keywords: Traveling salesman polytope, extended formulations, comb inequalities, lower bounds

1. Introduction

A polytope Q is called an extended formulation or an extension of polytope P if P can be obtained as a projection of Q. Extended formulations are of natural interest in combinatorial optimization because even if P has a large number of facets and vertices, there may exist a small extended formulation for it, allowing one to optimize a linear function over P indirectly by optimizing instead over Q. Indeed, many polytopes of interest admit small extended formulations. (See [1], for example, for a survey).

Recent years have seen many strong lower bounds on the size of extended formulations. In particular, Fiorini et al. [2] showed superpolynomial lower bounds for polytopes related to the MAX-CUT, TSP, and Independent Set problems. This was extended to more examples of polytopes related to other NP-hard problems having superpolynomial lower bounds [3, 4]. Even though these results are remarkable, they are hardly surprising since existence of a small extension for any of these polytopes would have extremly unexpected consequences in complexity theory.

Subsequently, Rothvoß showed that the perfect matching polytope of Edmonds does not admit a polynomial sized extended formulation [5], even though one can separate over it in polynomial time despite the polytope having exponentially many vertices and facets. To reconcile this apparent lack of power of compact extended formulations to capture even "easy" problems like perfect matching, the authors of this article introduced the notion of $\mathcal{H}$-free extended formulations [6].

Intuitively, in this setting, given a polytope P (presumably with a high extension complexity) and a set of valid inequalities $\mathcal{H}$, one would like to understand the extent to which the inequalities in $\mathcal{H}$ cause a bottleneck in finding a good extended formulation for P. More formally, the $\mathcal{H}$-free extension complexity of a polytope P measures the extension complexity of the polytope formed by removing the inequalities in $\mathcal{H}$ from the facet-defining inequalities of P. Particularly interesting classes of inequalities, for any polytope, are those for which one can construct an efficient separation oracle.

Clearly, in this setting, nothing interesting happens if the inequalities to be "removed" are redundant. In this article, we consider the Traveling Salesman Polytope and study its $\mathcal{H}$-free extension complexity when $\mathcal{H}$ is the set of simple comb inequalities or the set of 2-matching inequalities. Both sets of inequalities form important classes of inequalities for the TSP polytope. Whereas efficient separation algorithms are known for the 2-matching inequalities, no such algorithm is known for comb inequalities, which generalize the set of 2-matching inequalities [7, 8].

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In this article we identify a parameterized subset of comb inequalities which we call (h,t)*-uniform comb inequalities* where the parameters require a uniform intersection between the handle and all the teeth of the comb. We use these inequalities to show that the intersection of comb inequalities defines a polytope with exponential extension complexity. Furthermore we show that if $\mathcal{H}$ is a set of valid inequalities for the TSP polytope such that $\mathcal{H}$ does not contain the (h,t)-uniform comb inequalities for some values of parameters h and t, then the $\mathcal{H}$-free extension complexity of the TSP polytope on K_n is at least $2^{\Omega(n/t)}$. As corollaries we obtain exponential lower bounds for the $\mathcal{H}$-free extension complexity of the TSP polytope with respect to 2-matching inequalities and simple comb inequalities.

The rest of this article is organized as follows. In the next section we describe the comb and 2-matching inequalities and introduce the (h,t)-uniform comb inequalities. We also introduce the central tool that we use: subdivided prisms of graphs. After a brief motivation for the study of subdivided prisms in Section 3, we prove our main lemma in Section 4. We show that over suitably subdivided prisms of the complete graph, there exists a canonical way to translate perfect matchings into TSP tours that can be done without regard to any specific comb inequality. This translation, together with known tools developed in [9] connecting extension complexity with randomized communication protocols gives the desired results for the problems of interest. Finally, we discuss applications of the main result in Section 5.

2. Definitions

Let P be a polytope in $\mathbb{R}^d$. The extension complexity of P – denoted by $\mathrm{xc}(P)$ – is defined to be the smallest number r such that there exists an extended formulation Q of P with r facets.

Let $G=(V,E)$ be a graph. For any subset S of vertices, we denote the edges crossing the boundary of S by $\delta(S)$. That is, $\delta(S)$ denotes the set of edges $(u,v)\in E$ such that $|S\cap\{u,v\}|=1$.

The TSP polytope for the complete graph K_n is defined as the convex hull of the characteristic vectors of all TSP tours in K_n, and is denoted by TSP_n. Similary, PM_n denotes the convex hull of all perfect matchings in K_n. We say that any inequality $a^\intercal x\leqslant b$ is valid for a polytope P if every point in P satisfies this inequality. For a point v in P, the slack of v with respect to a valid inequality $a^\intercal x\leqslant b$ is defined to be the nonnegative number $b-a^\intercal v$.

2.1. Comb inequalities for TSP

For a graph $G=(V,E)$, a comb is defined by a subset of vertices H called the handle and a set of subsets of vertices $T_i, 1\leqslant i\leqslant k$ where k is an odd number at least three. The sets T_i are called the teeth. The handle and the teeth satisfy the following properties:

$$H\cap T_i\neq\emptyset, \tag{1}$$
$$T_i\cap T_j=\emptyset, \qquad \forall i\neq j \tag{2}$$
$$H\setminus\bigcup_{i=1}^{k}T_i\neq\emptyset \tag{3}$$

The following inequality is valid for the TSP polytope of G and is called the comb inequality for the comb defined by handle H and teeth T_i as above.

$$x(\delta(H))+\sum_{i=1}^{k}x(\delta(T_i))\geqslant 3k+1$$

Grötschel and Padberg [10] showed that every comb inequality defines a facet of TSP_n for each $n\geqslant 6$. It is not known whether separating over comb inequalities is NP-hard, neither is a polynomial time algorithm known.

For a given comb C and a TSP tour T of G, the slack between the corresponding comb inequality and T is denoted by $\mathrm{sl}_{\mathrm{comb}}(C,T)$.

2-matching inequalities

A comb inequality corresponding to a handle H and k teeth T_i is called a 2-matching inequality if each tooth T_i has size exactly two. In particular this means that $|H \cap T_i| = 1$ and $|T_i \setminus H| = 1$ for each $1 \leqslant i \leqslant k$. These inequalities are sometimes also referred to as blossom inequalities. Padberg and Rao [7] gave a polynomial time algorithm to separate over the 2-matching inequalities.

Simple comb inequalities

A comb inequality corresponding to a handle H and k teeth T_i is called a simple comb inequality if $|H \cap T_i| = 1$ or $|T_i \setminus H| = 1$ for each $1 \leqslant i \leqslant k$. Simple comb inequalities contain all the 2-matching inequalities. It is not known whether one can separate over them in polynomial time.

(h,t)-uniform comb inequalities

Let us define a subclass of comb inequalities called (h,t)-*uniform comb inequalities* associated with what we will call (h,t)-*uniform combs* for arbitrary $1 \leqslant h < t$. A comb, with handle H and k teeth T_i, is said be (h,t)-uniform if $|T_i| = t$ and $H \cap T_i = h$, for all $1 \leqslant i \leqslant k$.

2.2. Odd set inequalities for perfect matching

Let V denote the vertex set of K_n. A subset $U \subset V$ is called an odd set if the cardinality of U is odd. For every odd set U the following inequality is valid for the perfect matching polytope PM_n and is called an odd set inequality.

$$x(\delta(U)) \geqslant 1$$

For a given odd set S and a perfect matching M of K_n, the slack between the corresponding odd set inequality and M is denoted by $\mathrm{sl}_{\mathrm{odd}}(S, M)$.

2.3. t-subdivided prisms of a graph

A prism over a graph G is obtained by taking two copies of G and connecting corresponding vertices. It is helpful to visualise this as stacking the two copies one over the other and then connecting corresponding vertices in the two copies by a vertical edge. A t-subdivided prism is then obtained by subdividing the vertical edges by putting $t-2$ extra vertices on them. See Figure 1 for an example.

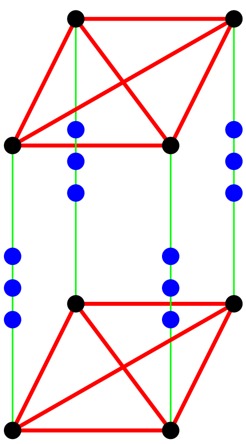


Figure 1: A 5-subdivided prism over K_4.

Let G be the t-subdivided prism of K_n. Let the vertices of the two copies be labeled $u_1^1, \dots, u_n^1$ and $u_1^t, \dots, u_n^t$. As a shorthand we will denote the path $u_i^1, u_i^2, \dots, u_i^t$ as $u_i^1 \rightsquigarrow u_i^t$. Similarily, $u_i^t \rightsquigarrow u_i^1$ will denote $u_i^t, \dots, u_i^2, u_i^1$.

The graph G has path $u_i^1 \rightsquigarrow u_i^t$ for all $i \in [n]$ and $(u_i^1, u_j^1), (u_i^t, u_j^t)$ for all $i \neq j, i, j \in [n]$. Thus G has tn vertices and $2\binom{n}{2} + (t-1)n$ edges.

3. Motivation

The motivation for looking at t-subdivided prisms stems from a simple observation which we state in the form of a proof of the following proposition:

Proposition 1. *Let $2MP(n)$ be the convex hull of the incidence vectors of all 2-matchings of the complete graph K_n. Then,* $\mathrm{xc}(2MP(n)) \geqslant 2^{\Omega(n)}$.

Proof. Let G be a graph with n vertices and m edges and let G' be the 3-subdivided prism of G. G' has $3n$ vertices and $2m + 2n$ edges. Any 2-matching in G' contains all the vertical edges and thus when restricted to a single copy – say the bottom one – of G gives a matching in G. Conversely, any matching in G can be extended to a (not necessarily unique) 2-matching in G'.

Taking G as K_n we obtain a G' that is a subgraph of K_{3n}. The 2-matching polytope of G' lies on a face of the 2-matching polytope of the complete graph on $3n$ vertices (corresponding to all missing edges having value 0). Therefore, the extension complexity of the 2-matching polytope $2MP(n)$ is at least as large as that of the perfect matching polytope. That is, $\mathrm{xc}(2MP(n)) \geqslant 2^{\Omega(n)}$. □

The above generalizes to p-matching polytopes for arbitrary p in the obvious way, and is probably part of folklore[1].

The generalization of the 3-subdivided prism to larger subdivisions allows us to be able to argue not only about the 2-matching inequalities – which are the facet-defining inequalities for the 2-matching polytope – but also about comb inequalities by using the vertical paths as teeth for constructing combs.

4. Main Tools

4.1. EF-protocols

Given a matrix M, a randomized communication protocol computing M in expectation is a protocol between two players Alice and Bob. The players, having full knowledge of the matrix M, agree upon some strategy. Next, Alice receives a row index i and Bob receives a column index j. Based on their agreed-upon strategy and their respective indices, they exchange a few bits and either one of them outputs a non-negative number, say X_{ij}. For brevity, we will call such protocols EF-protocols. An EF-protocol is said to correctly compute M if for every pair i, j of indices, $\mathbb{E}[X_{ij}] = M_{ij}$, where $\mathbb{E}[X_{ij}]$ is the expected value of the random variable X_{ij}.

The complexity of the protocol is measured by the number of bits exchanged by Alice and Bob in the worst case. It is known that the base-2 logarithm of the extension complexity of any polytope P is equal to the complexity of the best EF-protocol that correctly computes the slack matrix of P [9]. We will use this fact to show our lower bounds by showing that a sublinear EF-protocol for problems of our interest would yield a sublinear EF-protocol for the slack matrix of the perfect matching polytope. First we restate some known results about EF-protocols and extension complexity of perfect matching polytope in a language that will be readily usable to us.

Proposition 2. *[9] Let P be a polytope and $S(P)$ its slack matrix. There exists an EF protocol of complexity $\Theta(k)$ that correctly computes $S(P)$ if and only if there exists an extended formulation of P of size $2^{\Theta(k)}$.*

Combining lower bounds by Rothvoß [5] with the above mentioned equivalence by Faenza et al. [9], it is easy to see that no sublinear protocol computes the slack matrix of the perfect matching polytope.

Proposition 3. *[5] Any EF-protocol that correctly computes the slack matrix of the perfect matching polytope of K_n requires an exchange of $\Omega(n)$ bits.*

[1] W. Cook (private communication) attributes the same argument to T. Rothvoß

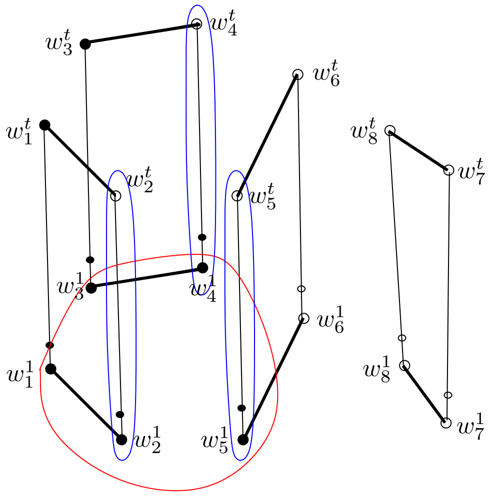


Figure 2: Construction of a comb from given odd set: The odd set consists of 5 vertices displayed as big filled circles in the bottom copy. The corresponding handle consists of all vertices represented by filled circles. The teeth are represented by the vertical ellipsoidal enclosures.
The big circles represent vertices of the original graph and their top copies. The small circles represent the h-th copy, while the other copies have been omitted here. Bold edges at the bottom are matching edges. All other edges displayed are just for illustration of the relationship of various copies of vertices.

4.2. Uniform combs of odd sets

Let n and t be positive integers. In the rest of the article we will assume that n is a multiple of t. Since we are interested in asymptotic statements only, this does not result in any loss of generality. Let G be the t-subdivided prism of $K_{n/t}$ for some $t \geqslant 2$. Given an odd set S and a perfect matching M in $K_{n/t}$, and arbitrary $1 \leqslant h < t$, we are interested in constructing a comb C and a TSP tour T in K_n such that the following conditions hold:

(C1): C is a (h,t)-uniform comb.

(C2): C depends only on S and 2 edges of M.

(C3): T depends only on M.

(C4): $\mathrm{sl}_{\mathrm{comb}}(C,T) = \mathrm{sl}_{\mathrm{odd}}(S,M)$.

If such a pair (C,T) of a comb and a TSP tour is shown to exist for every pair (S,M) of an odd set and a perfect matching, then we can show that any EF-protocol for computing the slack $\mathrm{sl}_{\mathrm{comb}}(C,T)$ can be used to construct an EF-protocol for computing $\mathrm{sl}_{\mathrm{odd}}(S,M)$ due to condition (C4). Furthermore, due to conditions (C2) and (C3) the number of bits required for the later protocol will not be much larger than the number of bits required for the former, as C can be locally constructed from S after an exchange of two edges, and T can be locally constructed from M.

Now we show that such a pair does exist if at least two edges of M are contained in S and $|S| \geqslant 5$.

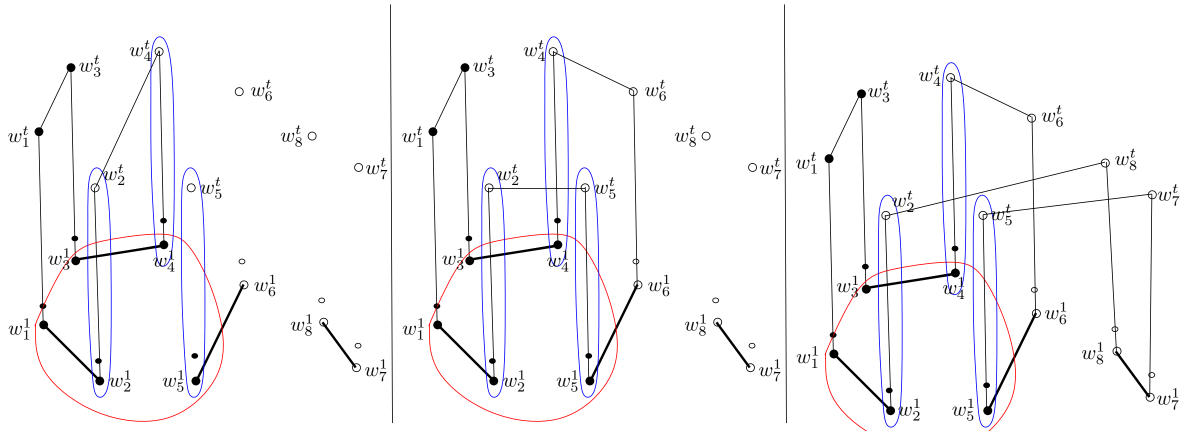
Lemma 1. *Let (S,M) be a pair of an odd set and a perfect matching in $K_{n/t}$, and let $1 \leqslant h < t$. Suppose that $|S| \geqslant 5$, and let $w_1, w_2, w_3, w_4 \in S$ be distinct with (w_1,w_2) and (w_3,w_4) in M. Then, there exists a pair (C,T) of a comb C and a TSP tour T in K_n satisfying the four conditions (C1)–(C4).*

Proof. Let $|S| = s$. For simplicity of exposition, we assume that the vertices of S are labeled $w_1, \ldots, w_s$. By w_i^j, we denote the copy of w_i in the j-th layer of the t-subdivided prism over $K_{n/t}$.

The comb C is constructed as follows. The handle H is obtained by taking all vertices in S and the copies $w_1^2, \ldots, w_1^t$ and $w_3^2, \ldots, w_3^t$. For every other vertex $w \in S$ the vertices $w^2, \ldots, w^h$ are also added to H. The teeth T_i are formed by pairing each vertex v in $S \setminus \{w_1, w_3\}$ with its copies $v^2, \ldots, v^t$ producing $s-2$ teeth. See Figure 2 for an illustration. Since $s \geqslant 5$ is odd, the number of teeth is odd and at least 3. Thus, the constructed comb is (h,t)-uniform satisfying conditions (C1) and (C2), and the corresponding comb inequality is

$$x(\delta(H)) + \sum_{i=1}^{s-2} x(\delta(T_i)) \geqslant 3(s-2) + 1. \tag{4}$$

To construct a tour T from the given perfect matching M such that conditions (C3) and (C4) are satisfied, we start with a subtour $(w_1^1 \rightsquigarrow w_1^t, w_3^t \rightsquigarrow w_3^1, w_4^1 \rightsquigarrow w_4^t, w_2^t \rightsquigarrow w_2^1, w_1^1)$. At each stage we maintain a



(a) The initial tour going through $w_1^1, w_1^t, w_3^t, w_3^1, w_4^1, w_4^t, w_2^t, w_2^1, w_1^1$ (b) Adding a new matching edge (w_5, w_6) by removing (w_2^t, w_4^t) (c) The final tour

Figure 3: Constructing a TSP tour from a perfect matching.

subtour that contains all matching edges on the induced vertices in the lower copy, the edge (w_1^t, w_3^t), and at least one top edge different from (w_1^t, w_3^t). Clearly the starting subtour satisfies these requirements. As long as we have some matching edges in M that are not in our subtour, we pick an arbitrary edge (w_a, w_b) in M and extend our subtour as follows. Select a top edge (w_q^t, w_r^t) different from (w_1^t, w_3^t), remove the edge and add the path $(w_q^t, w_a^t \rightsquigarrow w_a^1, w_b^1 \rightsquigarrow w_b^t, w_r^t)$. The new subtour contains the selected perfect matching edge (w_a^1, w_b^1), the paths $w_a^1 \rightsquigarrow w_a^t$ and $w_b^1 \rightsquigarrow w_b^t$ and has one more top edge distinct from (w_1^t, w_3^t) than in the previous subtour. See Figure 3 for an example.

At the completion of the procedure, we have a TSP tour that satisfies the following properties:

1. Each edge of M is used in the tour.
2. Each vertical path $w_i^1 \rightsquigarrow w_i^t$ for all $i \in [n]$ is used in the tour.
3. Edge (w_1^t, w_3^t) is used in the tour.

From the construction, edges in $|\delta(H) \cap T|$ are precisely the edges in $|\delta(S) \cap M|$ together with $s-2$ other edges exiting the comb: one through each of the $s-2$ teeth. Therefore, $|\delta(H) \cap T| = |\delta(S) \cap M| + s - 2$. Also, the tour T enters and exits each teeth precisely once so $|\delta(T_i) \cap T| = 2$ for each of the $s-2$ teeth. Substituting these values in the inequality 4, we obtain the slack $\mathrm{sl}_{\mathrm{comb}}(C, T) = |\delta(S) \cap M| + (s-2) + 2(s-2) - 3(s-2) - 1 = \mathrm{sl}_{\mathrm{odd}}(S, M)$. This completes the proof because the pair (C, T) satisfies conditions (C1)–(C4).

□

We are finally ready to state the main Lemma of this article. Using the existence of the pair (C, T) as described earlier and the fact that any EF-protocol for the perfect matching polytope requires an exchange of a linear number of bits, we will lower bound the number of bit exchanged by any EF-protocol computing the slack of (h, t)-uniform comb inequalities with respect to TSP tours. In the next Section we will use this Lemma multiple times by fixing different values for the parameters h and t.

Lemma 2. *Any EF-protocol computing the slack of (h, t)-uniform comb inequalities with respect to the TSP tours of K_n, requires an exchange of $\Omega(n/t)$ bits. Equivalently, the extension complexity of the polytope of (h, t)-uniform comb inequalities is $2^{\Omega(n/t)}$.*

Proof. Due to Proposition 3, it suffices to show if such a protocol uses r bits, then an EF-protocol for the perfect matching polytope for $K_{n/t}$ can be constructed, that uses $r + \mathcal{O}(\log(n/t))$ bits. The protocol for computing the slack of an odd set inequality with respect to a perfect matching in $K_{n/t}$ works as follows.

Suppose Alice has an odd set S in $K_{n/t}$, with $|S| = s$, and Bob has a matching M in $K_{n/t}$. The slack of the odd-set inequality corresponding to S with respect to matching M in the perfect matching polytope for $K_{n/t}$ is $|\delta(S) \cap M| - 1$.

We assume that $s \geqslant 5$. Otherwise, Alice can send the identity of the entire set S with at most $4 \log (n/t)$ bits and Bob can output the slack exactly.

Alice first sends an arbitrary vertex $w_1 \in S$, to Bob. Bob replies with the matching vertex of w_1, say w_2. Alice then sends another arbitrary vertex $w_3 \in S, w_3 \neq w_2$ to Bob who again replies with the matching vertex for w_3, say w_4. So far the number of bits exchanged is $4 \lceil \log (n/t) \rceil$.

Now there are two possibilities: either at least one of the vertices w_2, w_4 is not in S, or both w_2, w_4 are in S. Alice sends one bit to communicate which of the possibilities has occurred and accordingly they switch to one of the two protocols as described next.

In the former case, Alice has identified an edge, say e, in $\delta(S) \cap M$. Now Bob selects an edge e' of his matching uniformly at random (i.e. with probability $2/n$) and sends it to Alice. If e' is in $\delta(S) \setminus \{e\}$, Alice outputs $n/2$. Otherwise, Alice outputs zero. The expected contribution by edges in $(\delta(S) \cap M) \setminus \{e\}$ is then exactly one while the expected contribution of all other edges is zero. Therefore the expected output is $|\delta(S) \cap M| - 1$, and the number of bits exchanged for this step is $\lceil \log m \rceil$ where m is the number of edges in $K_{n/t}$. Thus the total cost in this case is $\mathcal{O}(\log (n/t))$ bits.

In the latter case, the matching edges (w_1, w_2) and (w_3, w_4) lie inside S. Alice constructs a comb C in the t-subdivided prism of $K_{n/t}$, and Bob a TSP tour T in the t-subdivided prism of $K_{n/t}$ such that (C, T) satisfies conditions (C1)–(C4). By Lemma 1 they can do this without exchanging any more bits. Since $\mathrm{sl}_{\,\mathrm{comb}}(C, T) = \mathrm{sl}_{\mathrm{odd}}(S, M)$, they proceed to compute the corresponding slack with the new inequality and tour, exchanging r bits. The total number of bits exchanged in this case is $r + 4 \lceil \log (n/t) \rceil + 1 = r + \mathcal{O}(\log (n/t))$. □

5. Applications

In this section we consider the extension complexity of the polytope of comb inequalities and $\mathcal{H}$-free extension complexity of the TSP polytope when $\mathcal{H}$ is the set of simple comb inequalities. As we will see, the results in this section are obtained by instantiating Lemma 2 with different values of the parameters h and t.

5.1. Extension complexity of Comb inequalities

We show that the polytope defined by the Comb inequalities has high extension complexity.

Theorem 1. *Let COMB(n) be the polytope defined by the intersection of all comb inequalities for* TSP_n*. Then* $\mathrm{xc}(\mathit{COMB}(n)) \geqslant 2^{\Omega(n)}$.

Proof. Suppose there exists an EF-protocol that computes the slack of COMB(n) that uses r bits. Since $(1, 2)$-uniform comb inequalities are valid for TSP_n we can use the given protocol to compute the slack of these inequalities with respect to the TSP tours of K_n using r bits. Then, using Lemma 2, the slack matrix of the perfect matching polytope for $K_{n/2}$ can be computed using $r + \mathcal{O}(\log n)$ bits. By Proposition 3, this must be $\Omega(n)$. Finally, by Proposition 2 this implies that $\mathrm{xc}(\mathrm{COMB}(n)) \geqslant 2^{\Omega(n)}$. □

5.2. $\mathcal{H}$-free extension complexity

Let $\mathcal{C}_{h,t}$ be the set of (h, t)-uniform comb inequalities for fixed values of h and t. Observe that, since at least three teeth are required to define a comb and the handle must contain some vertex not in any teeth, for (h, t)-uniform combs on n vertices we must have $t \leqslant \left\lfloor \frac{n-1}{3} \right\rfloor$. So for any values of $1 \leqslant h < t \leqslant \left\lfloor \frac{n-1}{3} \right\rfloor$, the set $\mathcal{C}_{h,t}$ is a nonempty set of facet-defining inequalities for TSP_n, and for any other values of h and t the set $\mathcal{C}_{h,t}$ is empty.

Theorem 2. *If $\mathcal{H}$ is a set of inequalities valid for the polytope* TSP_n*, such that $\mathcal{H} \cap \mathcal{C}_{h,t} = \emptyset$ for some nonempty $\mathcal{C}_{h,t}$, then the $\mathcal{H}$-free extension complexity of* TSP_n *is at least* $2^{\Omega(n/t)}$.

Proof. Let $1 \leqslant h < t$ be integers such that $\mathcal{H} \cap \mathcal{C}_{h,t} = \emptyset$. That is, the set $\mathcal{H}$ does not contain any (h, t)-uniform comb inequalities. Let P be the polytope formed from TSP_n by throwing away any facet-defining inequalities that are in $\mathcal{H}$. Then, any EF-protocol computing the slack matrix of P correctly must use $\Omega(n/t)$ bits due to Lemma 2. The claim then follows from Proposition 2. □

The above theorem shows that for every set $\mathcal{H}$ of valid inequalities of TSP_n, if the extension complexity of the TSP polytope becomes polynomial after removing the inequalities in $\mathcal{H}$, then $\mathcal{H}$ must contain some inequalities from every (h,t)-uniform comb inequality class, for all $t = o(n/\log n)$. The theorem can easily be made stronger by replacing the requirement $\mathcal{H} \cap \mathcal{C}_{h,t} = \emptyset$ with $|\mathcal{H} \cap \mathcal{C}_{h,t}| \leqslant \mathrm{poly}(n)$. (See the discussion about $\mathcal{H}$-free extension complexity of TSP_n with respect to subtour inequalities in Avis and Tiwary [6] for clarification.)

We can use the above theorem to give lower bounds for $\mathcal{H}$-free extension complexity of the TSP polytope with respect to important classes of valid inequalities by simply demonstrating some class of (h,t)-uniform comb inequalities that has been missed.

2-matching inequalities

Corollary 1. *Let P be the polytope obtained by removing the 2-matching inequalities from the TSP polytope. Then,* $\mathrm{xc}(P) = 2^{\Omega(n)}$.

Proof. The 2-matching inequalities are defined by combs for which each tooth has size exactly two. Therefore the set of $(1,3)$-uniform combs are not 2-matching inequalities, and Theorem 2 applies. □

Simple comb inequalities

Corollary 2. *Let P is the polytope obtained by removing the set of simple comb inequalities from the TSP polytope. Then,* $\mathrm{xc}(P) = 2^{\Omega(n)}$.

Proof. Recall that a comb is called simple if $|H \cap T_i| = 1$ or $|T_i \setminus H| = 1$ for all $1 \leqslant i \leqslant k$ where k is the (odd) number of teeth in the comb and H is the handle. Clearly, $(2,4)$-uniform combs are not simple and Theorem 2 applies. □

As mentioned before, simple comb inequalities define a superclass of 2-matching inequalities and a polynomial time separation algorithm is known for 2-matching inequalities. Althought a similar result was claimed for simple comb inequalities, the proof was apparently incorrect, as pointed out by Fleischer et al. [8]. This latter paper includes a polynomial time separation algorithm for the wider class of simple domino-parity inequalities that we do not consider here.

We leave as an open problem whether there exists a polynomial time separation algorithm for the (h,t)-uniform comb inequalities.

Acknowledgment

Research of the first author is supported by a Grant-in-Aid for Scientific Research on Innovative Areas – Exploring the Limits of Computation, MEXT, Japan. Research of the second author is partially supported by GA ČR grant P202-13/201414.

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